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Some relations between drag and flow pattern of viscous flow past a sphere and a cylinder at low and intermediate Reynolds numbers

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The results of a numerical evaluation of the Navier–Stokes equations of motion for the case of a viscous fluid streaming past a sphere are presented in terms of the length of the standing eddy behind the sphere and in terms of the angle of flow separation at the sphere. Emphasis was placed on calculating these quantities at Reynolds numbers between 20 and 40 where no reliable theoretical or experimental values are available. In support of these calculations, it is shown that the values for the drag on a sphere previously calculated by us from the Navier–Stokes equations of motion by the same numerical technique as that used for calculating the eddy length and angle of flow separation agree well with our recent, extensive drag measurements for a wide Reynolds number interval. Our results are used to make a comparison between drag and flow field as predicted by analytical solutions and numerical solutions to the Navier–Stokes equations of motion. Some limitations of the analytical solutions to predict correct values for the drag, and to describe the correct nature of the flow field, are pointed out. It is shown further that a plot of $[(D/D_s)-1]$ *versus* $\log N_{Re}$, where D is the actual drag on a sphere, D_s is the Stokes drag, and N_{Re} is the Reynolds number, reveals that the variation of the drag on a sphere with Reynolds number follows well defined régimes, which correlate well with the régimes of the flow field around a sphere. A similar relationship between 'drag-régime' and flow field pattern is discussed for the case of viscous flow past a cylinder.

1. Introduction

For over a century the sphere and the circular cylinder have been subject to numerous studies of viscous, incompressible flow at low and intermediate Reynolds numbers. During the last ten years, the problems related to viscous flow past a sphere and a cylinder have gained new and increasing attention. In particular, attention has been given to determining the drag on a liquid and rigid sphere by methods which involved solving the Navier–Stokes equation of motion analytically (Chester & Breach 1969; Proudman 1969; Ockendon 1968), or numerically (Le Clair, Hamielec & Pruppacher 1970; Rimon & Cheng 1969; Hamielec, Hoffman & Ross 1967; Hamielec & Johnson 1962; Hamielec, Storey & Whitehead 1962), and by experimental techniques (Beard & Pruppacher 1969; Le Clair *et al.* 1970; Pruppacher & Steinberger 1968; Goldburg & Florsheim 1966; Maxworthy 1965). Similar attention has been given to determining the drag on a circular cylinder by numerical methods (Underwood 1969; Hamielec & Raal 1969; Son & Hanratty 1969; Ingham 1968; Kawaguti & Jain 1966; Keller & Takami 1966; Dennis & Shimshoni 1964) and by experimental techniques (Jayaweera & Mason 1965; Tritton 1959). The interest in such drag studies has been stimulated by a variety of important problems in chemical engineering, air pollution, and cloud and precipitation physics.

In cloud physics, the problem of flow past spheres enters importantly in determining theoretically the growth by collision of cloud drops to rain drops in atmospheric clouds. As shown by Beard & Pruppacher (1969), cloud drops can well be approximated by rigid spheres up to $N_{Re} = 200$ (where $N_{Re} = 2aV_\infty/\nu$, a being the radius of the sphere, V_∞ the terminal velocity of the sphere and ν being the kinematic viscosity of the air), which is reasonable in the light of their large internal pressures, their extremely small deformation and their very small internal circulation (Pruppacher & Beard 1970). The problem of flow past cylinders enters importantly in determining theoretically the growth by collision of ice crystals the shape of which, in some cases, can roughly be approximated by a cylinder. The conditions for collision of two spherical or cylindrical shaped particles depends on an accurate description of the trajectory of the two interacting particles. This in turn makes necessary an accurate description of the flow field past these particles.

2. The sphere

Two quantities which characterize the flow field of a viscous fluid streaming past a sphere are the length of the standing eddy behind the sphere and the angle of flow separation at the sphere. Only a very few reliable experimental measurements and theoretical calculations are available on these two parameters, in particular at Reynolds numbers close to that at which the standing eddy starts to develop. We have therefore attempted to compute both of these quantities from a numerical solution of the steady state Navier–Stokes equations of motion for an incompressible fluid. Since the physical model and the computational technique have been described in detail in two earlier papers in conjunction with our calculations of the drag on a sphere (Hamielec *et al.* 1967; Le Clair *et al.* 1970), we shall merely present and discuss our results here.

Our results for the variation of the eddy length L/d as a function of N_{Re} are presented in figure 1, and our results for the variation of the angle of flow separation θ_s as a function of $\log N_{Re}$ are presented in figure 2, where they are compared with the theoretical results of Jenson (1959), Hamielec *et al.* (1967), Rimon & Cheng (1969), Rhodes (1967), and Proudman & Pearson (1957), and with the experimental results of Taneda (1956*a*), Garner & Grafton (1954), and Nisi & Porter (1923). It is seen from these figures that the agreement between Taneda's experimental and our theoretical values is good, except at low Reynolds numbers where Taneda somewhat underestimates the eddy length and angle of separation. This discrepancy is expected if we consider the extreme difficulties in visualizing

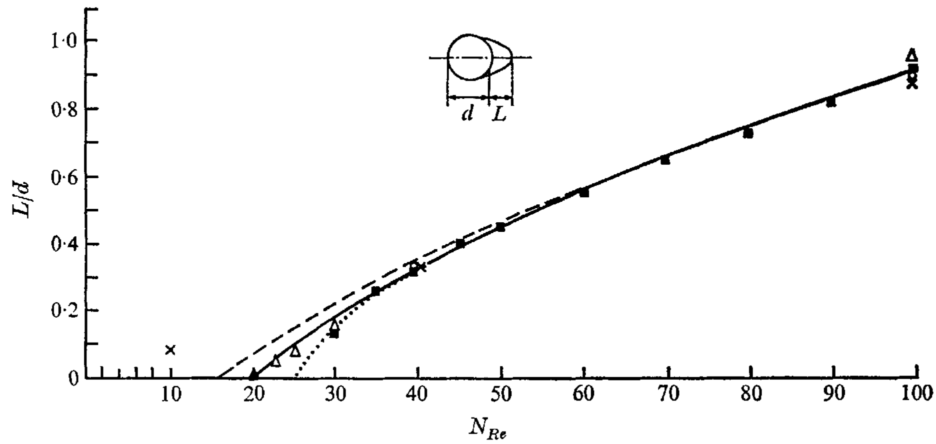

FIGURE 1. Variation of the eddy length with the Reynolds number for the case of a sphere. Theory: ○, Hamielec–Hoffman–Ross; ×, Rimon–Cheng; △, present results; – – –, Proudman–Pearson (from first 2 terms of the Stokes expansion); ——, present proposed variation of L/d with N_{Re}. Experiment: ...■..., Taneda.

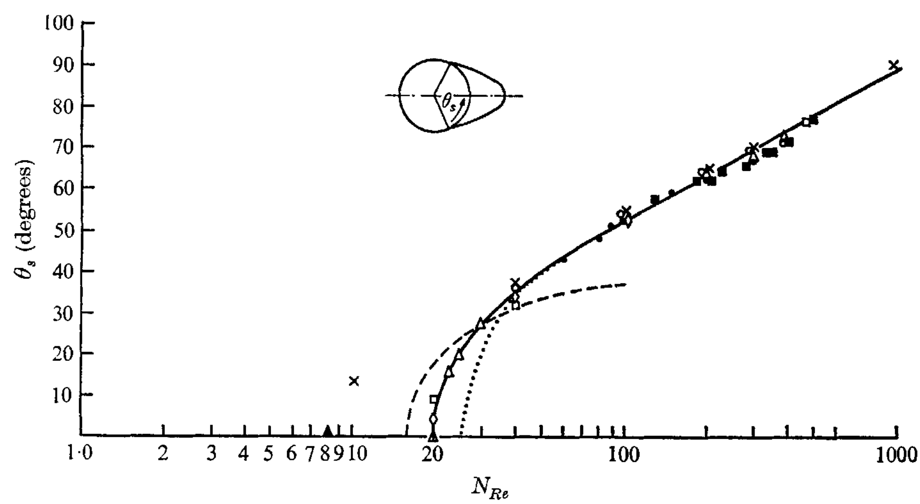

FIGURE 2. Variation of the angle of flow separation with Reynolds number for the case of a sphere. Theory: □, Jenson; ○, Hamielec–Hoffman–Ross; ×, Rimon–Cheng; ◇, Rhodes; △, present results; – – –, Proudman–Pearson (from first 2 terms of the Stokes expansion); ——, present proposed variation of θ_s with N_{Re}.

the flow at small Reynolds numbers with tracer particles of finite size and finite fall velocities relative to the fluid motion. The experimental results of Nisi & Porter were probably biased by their method of sphere support. The value for L/d, obtained theoretically by Rimon & Cheng for $N_{Re} = 10$, is probably not valid for a sphere in undisturbed, parallel flow, as their boundary conditions were satisfied too close to the sphere surface. Our numerical computations (displayed in figures 1 and 2) indicate that onset of the development of a standing eddy in the back of the sphere takes place at $N_{Re} = 20$. By varying step size and distance to the outer boundary, we made sure that our computed values for L/d and θ_s were not biased by these parameters. The effect of step size and wall proximity on θ_s for $N_{Re} = 20$ is shown in table 1. This table suggests that indeed $\theta_s = 0$ for $N_{Re} = 20$. Our values for L/d and θ_s are further compared with those determined by Proudman & Pearson from a stream function based on two terms of the Stokes expansion. These relations predict an onset of a recirculatory wake at $N_{Re} = 16$, and a growth of the eddy, which is in astonishingly good agreement with the numerical results as far as its length is concerned, but is in quite poor agreement with those as far as its angular extent is concerned. In case the next

	Δz	$\Delta\theta$ (°)	r_∞	θ_s (°)
Jenson (1959)	0·1	12	6·05	9
Rhodes (1967)	0·05	6	6·05	2·5
Hamielec *et al.* (1967)	0·05	3	7·02	0
Present results	0·05	3	90	0
	0·0167	3	90	0

TABLE 1. Effect of step size on the computation of angle of flow separation at $N_{Re} = 20$ for the case of a sphere; Δz is the radial step size, $\Delta\theta$ is the angular step size and $r_\infty = r'_\infty/a$ is the non-dimensional distance to the boundary

higher-order term in the Stokes expansion of Proudman & Pearson is taken into account, no eddy at all is predicted (Van Dyke 1964). This result is very surprising, but it is consistent with the findings of Chester & Breach (1969) regarding the drag discussed below. Unfortunately, available theoretical models and numerical techniques have up to the present not allowed the prediction of the Reynolds number for onset of shedding of vortices from the rear of a sphere. Experiments by Möller (1938), Taneda (1956*a*) and Goldburg & Florsheim (1966) indicate that shedding of vortices begins at Reynolds numbers between 300 and 450.

In support of the calculations given above, we summarized in figure 3 our recent data (Hamielec *et al.* 1967; Le Chair *et al.* 1970) for the drag on a sphere obtained from solving the Navier–Stokes equations by the same numerical method used for the calculations of the eddy length and angle of flow separation. These theoretical results are compared with our recent experimental determination of the drag determined on solid spheres falling in oil (Pruppacher & Steinberger 1968; Le Clair *et al.* 1970), and on small water drops (Beard & Pruppacher 1969) freely floating in air of a large, vertical wind tunnel, described by Pruppacher & Neiburger (1968). The good agreement between our experimental results and those given by Perry (1950) for Reynolds numbers between 70 and 300 motivated our use of Perry's data for extending the experimental drag curve up to $N_{Re} = 5000$. Furthermore, we thought it may be instructive to compare our theoretical and experimental values for the drag on a sphere with the numerical results of Jenson (1959), Rimon & Cheng (1969) and with the analytical results of Oseen (1927), Goldstein (1929), Carrier (1953), Proudman & Pearson (1957), and in particular with the results of Chester & Breach (1969) and Proudman (1969). From figure 3, where the variation of $\log[(D/D_s)-1]$ with $\log N_{Re}$ is

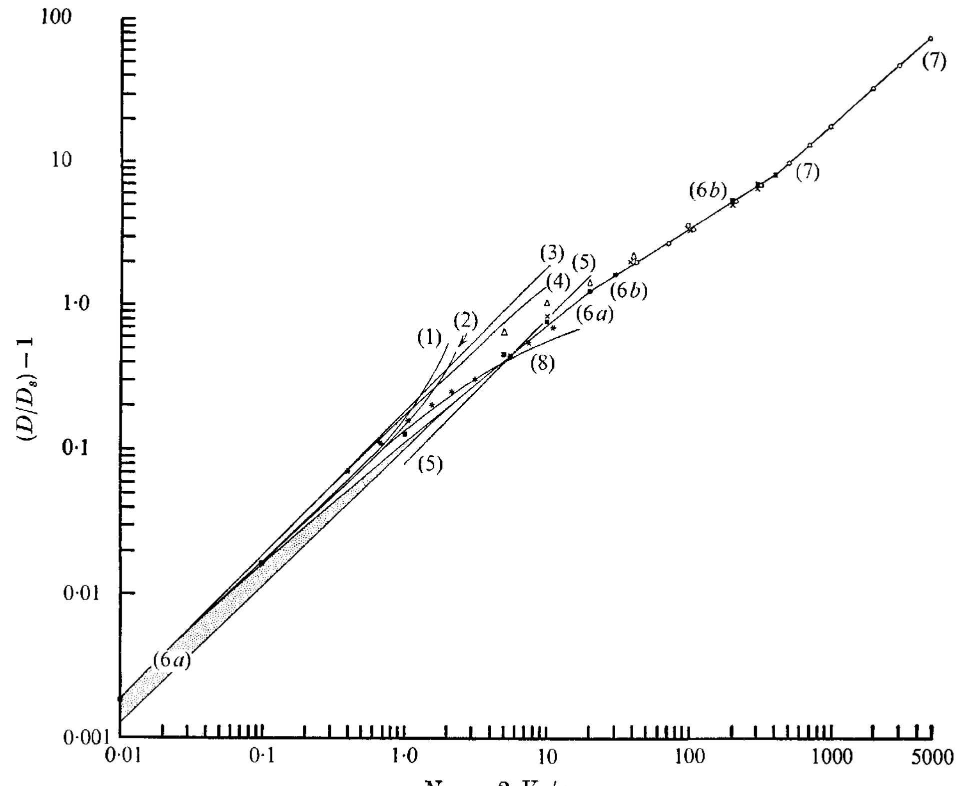

FIGURE 3. Variation of the quantity $(D/D_s)-1$ with the Reynolds number for the case of a sphere: $*$, Maxworthy (experimental); ■, Le Clair–Hamielec–Pruppacher; □, Hamielec–Hoffman–Ross; △, Jenson; ×, Rimon–Cheng; (1), Chester–Breach; (2), Proudman–Pearson; (3), Oseen; (4), Goldstein; (5), Carrier; (6a)–(6b), Pruppacher, Beard–Pruppacher, Pruppacher–Steinberger (experimental); ○, (7), Perry (experimental); (8), Proudman ($m = 5$); ▦, experimental scatter.

plotted, D being the actual drag and D_s the Stokes drag, the following conclusions are suggested:

(i) For a wide range of Reynolds numbers $0{\cdot}01 \leqslant N_{Re} \leqslant 400$, the drag on a sphere computed from our numerical solution of the Navier–Stokes equations of motion agrees well with our experimental values. However, extreme care has to be taken to correct errors introduced by step size and wall effects, in particular, at low Reynolds numbers, as pointed out by Hamielec *et al.* (1967), and Le Clair *et al.* (1970). The results of Jenson, plotted in figure 3, are an example of results containing errors which were introduced by such effects. The success of the numerical approach is contrasted in figure 3 by the results of analytical solutions to the Navier–Stokes equations of motion by Oseen (1927), Goldstein (1968), and Proudman & Pearson (1957), which satisfactorily represent the drag on a sphere only up to about $N_{Re} = 0{\cdot}1$. At larger Reynolds numbers the analytical results depart increasingly from the true drag.

It is of particular interest to note that, surprisingly enough, this situation is not altered by the presently available refinements of the analytical expressions. It is seen from figure 3 that the recent results of Chester & Breach (1969), who extended the Proudman & Pearson theory to higher-order terms, do not give a better fit to the actual drag. Proudman (1969) tried to remedy this problem by pointing out that the poor convergence of the Proudman & Pearson expansion is, at least in part, due to the lack of suitability of the function D for expansion in terms of N_{Re}. Based on semi-empirical arguments, Proudman suggested a new functional relationship between N_{Re} and D/D_s, which we have evaluated numerically for various values of the parameter m in his relationship. We plotted the results for the case that $m = 5$. Smaller values of m give a progressively worse fit to our experimental drag at low Reynolds numbers and a fit which is not much better at higher Reynolds numbers. Larger values of m do not improve the fit at low Reynolds numbers, and make the bad fit at higher Reynolds numbers even worse. It is seen from figure 3 that Proudman's (1969) proposed semi-empirical relationship provides indeed a substantial improvement over that proposed by Proudman & Pearson (1957) up to $N_{Re} \approx 7$. At larger Reynolds numbers, Proudman's results deviate sharply from the actual drag. At Reynolds numbers between 1 and 7, the fit to the actual drag is about as good as that of Carrier's (1953) semi-empirical modification of the Oseen theory. The agreement between our experimental values for the drag and the experimental values of Maxworthy (1965) is satisfactory at Reynolds numbers between 3 and 12. At $N_{Re} < 3$ Maxworthy's values deviate progressively from ours reaching the Oseen drag at a Reynolds number of about $0{\cdot}4$, while our experimental and theoretical values for the drag approach the Oseen drag via the Proudman–Pearson drag.

It is felt that the good agreement between our theoretically obtained drag and our experimentally determined drag give support to our values for the eddy length and angle of flow separation, which were calculated on the basis of the same physical and numerical model as the drag.

(ii) Another conclusion can also be drawn from figure 3. A close inspection of our drag curve reveals that the variation of the drag with Reynolds number follows well-defined régimes, which are quite pronounced in our experimental drag curve, and slightly less pronounced by our theoretical drag values. These régimes are manifested by almost constant slopes of the drag curve within certain Reynolds number intervals, and rather strong changes of slope at a particular Reynolds number. Two Reynolds numbers at which a change of régime occurs were noted. The first change of 'drag régime' appears to lie at $N_{Re} = 20$, which coincides with the Reynolds number at which our numerical results indicate onset of separation and formation of a standing vortex in the back of the sphere. The second change of régime seems to occur at $N_{Re} = 400$, which coincides with the Reynolds number at which experiments reported in literature indicate onset of shedding of vortices from the downstream end of the sphere into the

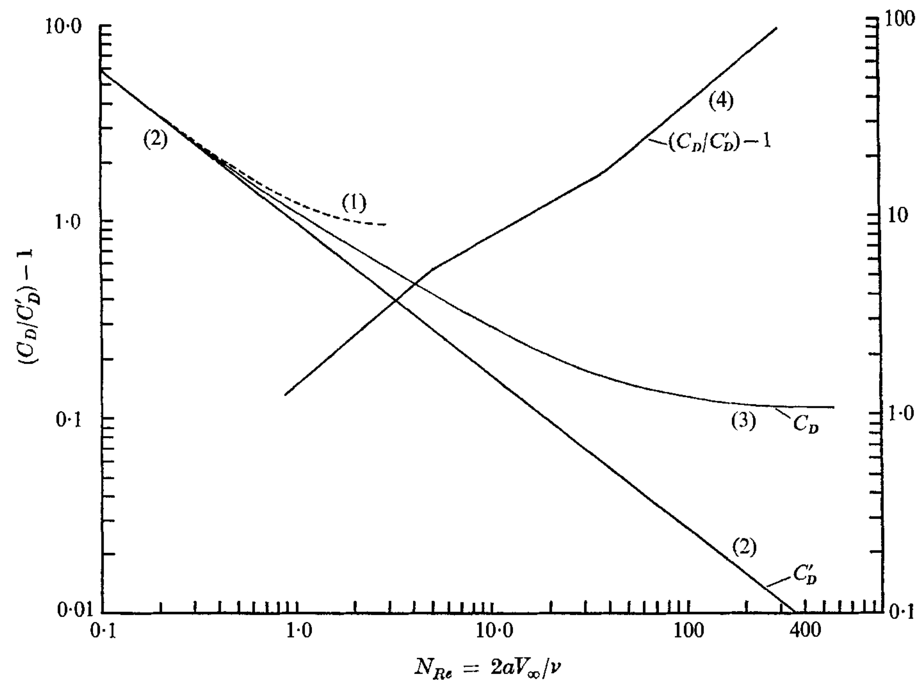

FIGURE 4. Variation of the drag force coefficient and the relative drag force coefficient with Reynolds number for the case of a cylinder: (1), Lamb's formula; (2), asymptote to Lamb's formula at $N_{Re} = 0{\cdot}1$; (3), curve fitted to experimental data from literature; (4), C_D/C'_D obtained graphically.

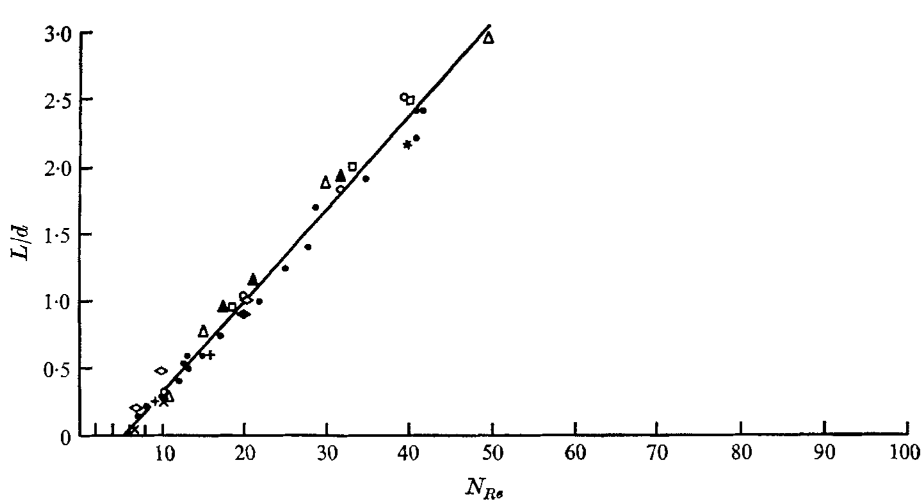

FIGURE 5. Variation of the eddy length with Reynolds number for the case of a cylinder. Theory: ○, Kawaguti–Jain; ×, Underwood; □, Son–Hanratty; △, Hamielec–Raal; *, Apelt; +, Keller–Takami; ◆, Thom; ◇, Dennis–Shimshoni. Experiment: ■, Homann; ▲, Fage; ●, Taneda.

main stream. This indicates that the régimes known to be present in the flow field induce régimes in the drag. This feature is somewhat expected, but could only be revealed by accurate values of the drag if plotted on a sensitive scale such as $\log[(D/D_s)-1]$ *versus* $\log N_{Re}$. Such drag régimes are a very useful feature of the drag, since it is possible to approximate the drag in each régime by an empirical relation of the form $(D/D_s)-1 = \alpha N_{Re}^{\beta}$, where α and β are constants, a relationship used by Beard & Pruppacher (1969) to compute the fall velocity of cloud drops for any level in the atmosphere.

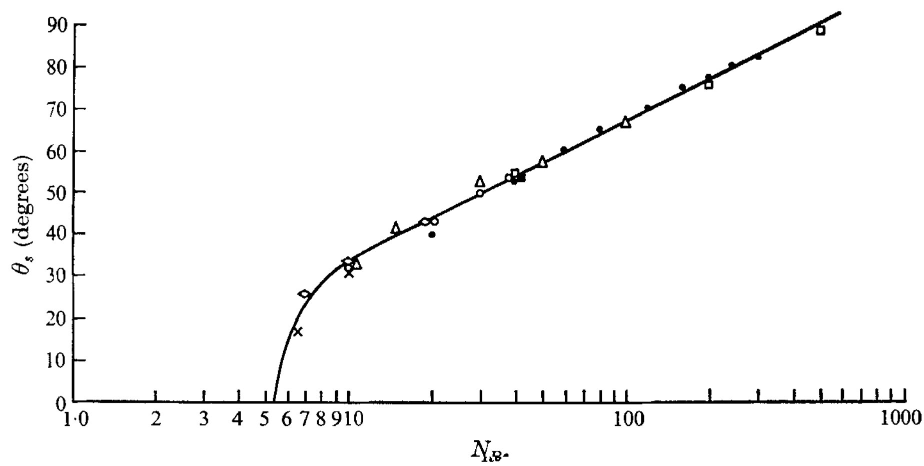

FIGURE 6. Variation of the angle of flow separation with Reynolds number for the case of a cylinder. Theory: ○, Kawaguti–Jain; ×, Underwood; □, Son–Hanratty; △, Hamielec–Raal; ◇, Dennis-Shimshoni. Experiment: ●, Homann; ■, Taneda.

3. The cylinder

It was tempting to test whether the relations between drag and flow field found for the case of flow past a sphere would also be found for the case of flow past a circular cylinder. We compiled the best measurements available for the drag coefficient C_D of a cylinder (Wieselberger 1922; Finn 1953; Tritton 1959; Jayaweera & Mason 1965), plotted their results in form of $\log C_D$ as a function of $\log N_{Re}$ and fitted a curve to these values with great care. This curve is given in figure 4. (Due to photographic reduction in size and reproduction for printing the curve given in this article is only a qualitative, and not a quantitative, reproduction of the large detailed original graph!) For comparison, the variation of the drag coefficient with the Reynolds number given by the formula of Lamb (1932) is also given. Completely arbitrarily we choose to draw an asymptote to Lamb's drag curve at $N_{Re} = 0{\cdot}1$, in order to obtain a reference drag coefficient C'_D. Graphically, values for $(C_D/C'_D)-1$ were determined for $1{\cdot}0 \leqslant N_{Re} \leqslant 300$ and plotted on log-log scale in figure 4. It is seen from this figure that the variation with Reynolds number of $(C_D/C'_D)-1$ follows pronounced régimes. Two Reynolds numbers at which a change of régime occurs were noted. The first change of régime appears to be at $N_{Re} = 5$, and the second at $N_{Re} = 40$. In order to test whether these two changes of régimes in the drag curve are related to the flow field régimes around the cylinder, we plotted in figures 5 and 6 values for the length of the standing eddy and the angle of flow separation at a cylinder based on the theoretical values of Kawaguti & Jain (1966), Underwood (1969), Son & Hanratty (1969), Hamielec & Raal (1969), Apelt (1958), Keller & Takami (1966), Thom (1933), Dennis & Shimshoni (1964), and based on the experiments of Taneda (1956*b*), Fage (1934), and Homann (1936). The variation of L/d and θ_s with Reynolds number indicates that a standing eddy begins to develop at $N_{Re} \approx 5$. Experiments carried out by Taneda (1956*b*), Roshko (1954) and Homann indicate that vortices from the rear of the cylinder begin to be shed at $N_{Re} \approx 40$. The agreement between the Reynolds numbers marking the transition between flow field régimes with the Reynolds numbers marking the changes in the drag régimes demonstrates that, as in the case of a sphere, also in the case of a cylinder the drag field régimes reflect the régimes in the flow field.

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